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Nonlinear differential–difference equations, associated Bäcklund transformations and Lax technique

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Abstract. The authors show that the Lax technique can be successfully employed to derive the class of nonlinear differential–difference equations associated with the discrete analogue of the matrix Schrödinger spectral problem, and solvable by the spectral transform.

1. Introduction

In two previous papers (Bruschi and Ragnisco 1980a, b) we have shown that the Lax technique allows us to recover the whole class of nonlinear evolution equations (NEEs) and Bäcklund transformations (BTs) associated with the matrix Schrödinger spectral problem (Wadati and Kamijo 1974), already derived by Calogero and Degasperis (1977) through the generalised Wronskian technique. In the present paper we prove that the equivalence between Lax and generalised Wronskian technique holds true also for the nonlinear differential–difference equations (NDDEs), and the corresponding Bäcklund transformations associated with the discrete analogue of the matrix Schrödinger spectral problem (Bruschi *et al* 1980, 1981). In § 2 we derive, through the Lax technique, the class of NDDEs, while in § 3 we deal, in the same framework, with BTs. All proofs are confined to § 4.

2. NDDEs and Lax technique

Let us consider the eigenvalue problem

$$L\boldsymbol{\psi}(n, t; z) = \lambda\boldsymbol{\psi}(n, t; z) \qquad (\lambda = z + z^{-1}) \tag{1}$$

where, throughout this paper,

$$L = E^{-} + \boldsymbol{B}(n, t)I + \boldsymbol{A}(n, t)E^{+}, \tag{2}$$

$E^{\pm}$ being the usual shift operators, $E^{\pm}\boldsymbol{\psi}(n, t; z) = \boldsymbol{\psi}(n \pm 1, t; z)$, and $\boldsymbol{A}(n, t)$, $\boldsymbol{B}(n, t)$ being two $N \times N$ matrices, depending on the integer variable n and (parametrically) on the real variable t, obeying the asymptotic conditions

$$\lim_{|n|\to\infty} \boldsymbol{A}(n, t) - \boldsymbol{I} = 0, \qquad \lim_{|n|\to\infty} \boldsymbol{B}(n, t) = 0. \tag{3}$$

In formulae (2) and (3) $\boldsymbol{I}$ denotes, as usual, the identity operator. It can be shown

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(Bruschi *et al* 1980) that, provided the 'potentials' $\boldsymbol{A}(n, t)-\boldsymbol{I}$ and $\boldsymbol{B}(n, t)$ decrease sufficiently fast as $|n| \rightarrow \infty$, the spectrum of L consists of two disjoint sets: the continuous spectrum, given by the unit circle in the z complex plane, and the discrete spectrum, given by a finite number of points all lying inside the unit circle. Analogously to the continuous case, it is possible to prove the following lemma.

Lemma 1. Let L be a spectral operator, and let there exist an operator M such that the commutator $[L, M]$ takes the form

$$[L, M]=\boldsymbol{Q}(n, t) I+\boldsymbol{P}(n, t) E^{+}.$$

Then (by a dot we denote t-derivative)

$$\dot{L}=[L, M] \tag{4}$$

defines a system of evolution equations for the 'potentials' $\boldsymbol{A}(n, t), \boldsymbol{B}(n, t)$ such that (i) the spectrum of L does not change with t and (ii) the eigenfunction $\boldsymbol{\psi}$ evolves according to the formula

$$\dot{\boldsymbol{\psi}}(n, t ; z)=-M \boldsymbol{\psi}(n, t ; z)+\boldsymbol{\psi}(n, t ; z) \boldsymbol{\chi}(z, t),$$

the t-dependent $N \times N$ matrix $\boldsymbol{\chi}$ being determined by the asymptotic behaviour of $\boldsymbol{\psi}$.

To derive the class of evolution equations associated with the spectral problem (1), the crucial step is given by the following theorem.

Theorem 1. Let there exist an operator M such that

$$[L, M]=\boldsymbol{Q}(n, t) I+\boldsymbol{P}(n, t) E^{+}$$

where the $N \times N$ matrices $\boldsymbol{Q}(n, t), \boldsymbol{P}(n, t)$ depend of course in general upon the 'potentials'. Define the new operator

$$M^{\prime}=O M+\bar{M} \tag{5}$$

where

$$\bar{M}=(\boldsymbol{\Pi}(n, t) \overline{\boldsymbol{F}}_{1}(t) \boldsymbol{\Pi}^{-1}(n+1, t)) E^{+}+\overline{\boldsymbol{F}}_{2}(t) I, \tag{6}$$

$\overline{\boldsymbol{F}}_{1}, \overline{\boldsymbol{F}}_{2}$ being two n-independent matrices and $\boldsymbol{\Pi}(n, t)$ being defined as

$$\boldsymbol{\Pi}(n, t)=\boldsymbol{A}(n, t) \boldsymbol{A}(n+1, t) \cdots=\prod_{j=n}^{\infty} \boldsymbol{A}(j, t),$$

and the operator O is defined by

$$O M=L M+\boldsymbol{F}_{1} E^{+}+\boldsymbol{F}_{2} I \tag{7}$$

where

$$\begin{aligned}
\boldsymbol{F}_{1}(n, t) & =\boldsymbol{\Pi}(n, t) \sum_{j=n+1}^{\infty}(\boldsymbol{\Pi}(j, t))^{-1} \boldsymbol{P}(j, t) \boldsymbol{\Pi}(j+1, t)(\boldsymbol{\Pi}(n+1, t))^{-1} \\
& =\boldsymbol{A}(n, t) \boldsymbol{S}(n+1, t)(\boldsymbol{A}(n+1, t))^{-1}
\end{aligned}$$

and

$$\boldsymbol{F}_{2}(n, t)=-\sum_{j=n}^{\infty} \boldsymbol{Q}(j, t).$$

Then it follows that

$$[L, M'] = \boldsymbol{Q}'(n, t)I + \boldsymbol{P}'(n, t)E^{+}, \tag{8}$$

the $2N \times N$ matrix $\begin{bmatrix} \boldsymbol{P}' \\ \boldsymbol{Q}' \end{bmatrix}$ being related to $\begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{bmatrix}$ through the formula

$$\begin{bmatrix} \boldsymbol{P}'(n, t) \\ \boldsymbol{Q}'(n, t) \end{bmatrix} = \mathscr{L}\begin{bmatrix} \boldsymbol{P}(n, t) \\ \boldsymbol{Q}(n, t) \end{bmatrix} + \begin{bmatrix} \bar{\boldsymbol{P}}(n, t) \\ \bar{\boldsymbol{Q}}(n, t) \end{bmatrix} \tag{9}$$

where

$$\mathscr{L}\begin{bmatrix} \boldsymbol{P}(n, t) \\ \boldsymbol{Q}(n, t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{P}(n, t)\boldsymbol{B}(n+1, t) + \boldsymbol{A}(n, t)(\boldsymbol{Q}(n, t) + \boldsymbol{Q}(n+1, t)) + \boldsymbol{B}(n, t)\boldsymbol{S}(n, t) - \boldsymbol{S}(n, t)\boldsymbol{B}(n+1, t) + \sum_{j=n}^{\infty} [\boldsymbol{P}(j, t), \boldsymbol{A}(n, t)] \\ \boldsymbol{B}(n, t)\boldsymbol{Q}(n, t) + \boldsymbol{P}(n, t) + \boldsymbol{S}(n-1, t) - \boldsymbol{S}(n, t) + \sum_{j=n}^{\infty} [\boldsymbol{Q}(j, t), \boldsymbol{B}(n, t)] \end{bmatrix}, \tag{10a}$$

$$\begin{aligned} \bar{\boldsymbol{P}}(n, t) &= [\boldsymbol{A}(n, t), \bar{\boldsymbol{F}}_2(t)] + \boldsymbol{B}(n, t)\boldsymbol{\Pi}(n, t)\bar{\boldsymbol{F}}_1(t)\boldsymbol{\Pi}^{-1}(n+1, t) \\ &\quad - \boldsymbol{\Pi}(n, t)\bar{\boldsymbol{F}}_1(t)\boldsymbol{\Pi}^{-1}(n+1, t)\boldsymbol{B}(n+1, t), \\ \bar{\boldsymbol{Q}}(n, t) &= [\boldsymbol{B}(n, t), \bar{\boldsymbol{F}}_2(t)] + \boldsymbol{\Pi}(n-1, t)\bar{\boldsymbol{F}}_1(t)\boldsymbol{\Pi}^{-1}(n, t) - \boldsymbol{\Pi}(n, t)\bar{\boldsymbol{F}}_1(t)\boldsymbol{\Pi}^{-1}(n+1, t). \end{aligned} \tag{10b}$$

Note that the operator $\mathscr{L}$ defined by (10*a*) is just the operator $\boldsymbol{L}$ derived by Bruschi *et al* (1981).

Introducing now, in the space of $N \times N$ matrices, the basis $\{\boldsymbol{\sigma}_\nu\}_{\nu=0,\dots,N^2-1}(\boldsymbol{\sigma}_0 = I)$ enables us to give the main result of this section, expressed by the next theorem.

Theorem 2. Let the operator M be defined through the formula

$$M = f_{1,\nu}(O, t)\boldsymbol{\Pi}(n,t)\boldsymbol{\sigma}_\nu\boldsymbol{\Pi}^{-1}(n+1, t)E^{+} + f_{2,\nu}(O, t)\boldsymbol{\sigma}_\nu I \tag{11}$$

where $f_{1,\nu}, f_{2,\nu}$ are arbitrary entire functions of their first argument and the operator O is defined by (7).

Then the Lax equation (5) implies that the 'potentials' evolve according to the system of NDDEs

$$\begin{pmatrix} \dot{\boldsymbol{A}}(n, t) \\ \dot{\boldsymbol{B}}(n, t) \end{pmatrix} + f_{1,\nu}(\mathscr{L}, t)\begin{pmatrix} \boldsymbol{B}(n, t)\boldsymbol{\Pi}(n, t)\boldsymbol{\sigma}_\nu\boldsymbol{\Pi}^{-1}(n+1, t) - \boldsymbol{\Pi}(n, t)\boldsymbol{\sigma}_\nu\boldsymbol{\Pi}^{-1}(n+1, t)\boldsymbol{B}(n+1, t) \\ \boldsymbol{\Pi}(n-1, t)\boldsymbol{\sigma}_\nu\boldsymbol{\Pi}^{-1}(n, t) - \boldsymbol{\Pi}(n, t)\boldsymbol{\sigma}_\nu\boldsymbol{\Pi}^{-1}(n+1, t) \end{pmatrix}$$
$$+ f_{2,\nu}(\mathscr{L}, t)\begin{pmatrix} [\boldsymbol{A}(n, t), \boldsymbol{\sigma}_\nu] \\ [\boldsymbol{B}(n, t), \boldsymbol{\sigma}_\nu] \end{pmatrix}$$

while the spectrum of L remains unchanged and the eigenfunction $\boldsymbol{\psi}$ satisfies the following evolution equation:

$$\dot{\boldsymbol{\psi}}(n, t; z) = -M\boldsymbol{\psi}(n, t; z) + \boldsymbol{\psi}(n, t; z)\boldsymbol{\chi}(z, t). \tag{12}$$

From the previous theorem one can easily derive the following corollary.

Corollary 2.1. Let the eigenfunction $\boldsymbol{\psi}$ evolve according to the equation (12) and be characterised by the asymptotic behaviour

$$\boldsymbol{\psi}(n, t; z) \underset{n\to+\infty}{\sim} z^{-n} + \boldsymbol{R}(z, t)z^{n}, \qquad \boldsymbol{\psi}(n, t; z) \underset{n\to-\infty}{\sim} z^{-n}\boldsymbol{T}(z, t).$$

Then

$$\boldsymbol{\chi}(z, t) = (f_{1,\nu}(\lambda, t)z^{-1} + f_{2,\nu}(\lambda, t))\boldsymbol{\sigma}_{\nu}$$

and

$$\dot{\boldsymbol{R}}(z, t) = (\tfrac{1}{2}\lambda f_{1,\nu}(\lambda, t) + f_{2,\nu}(\lambda, t))[\boldsymbol{R}(z, t), \boldsymbol{\sigma}_{\nu}] + \tfrac{1}{2}\mu f_{1,\nu}(\lambda, t)\{\boldsymbol{R}(z, t), \boldsymbol{\sigma}_{\nu}\} \tag{13}$$

where we have set

$$\lambda = z + z^{-1}, \qquad \mu = z^{-1} - z.$$

Note that the evolution of the reflection coefficient $\boldsymbol{R}(z, t)$ (13) is nothing but the one given by Bruschi *et al* (1981), any difference being merely notational.

2. BTs and Lax technique

To derive, by a proper generalisation of the Lax technique, the BTs related to the class of NDDEs derived in the previous section, first of all we have to consider the two eigenvalue problems

$$L\boldsymbol{\psi} = (z + z^{-1})\boldsymbol{\psi}, \tag{14a}$$

$$L'\boldsymbol{\psi}' = (z + z^{-1})\boldsymbol{\psi}', \tag{14b}$$

where L' is again defined by formula (2), once we replace the pair $(\boldsymbol{A}(n, t), \boldsymbol{B}(n, t))$ by the pair $(\boldsymbol{A}'(n, t), \boldsymbol{B}'(n, t))$, which also exhibits the asymptotic behaviour (3), so that the continuous spectra of L and L' are the same, being both given by the unit circle in the z complex plane.

A preliminary step is accomplished via the following lemma.

Lemma 2. Let H denote an operator and $\boldsymbol{\varphi}$ an $N \times N$ t-dependent matrix. Then the operators L, L', H are related by the formula

$$L'H - HL = 0 \tag{15}$$

if and only if the corresponding eigenfunctions satisfy the relation

$$\boldsymbol{\psi}'\boldsymbol{\varphi} - H\boldsymbol{\psi} = 0, \tag{16}$$

$\boldsymbol{\varphi}$ being determined by the asymptotic behaviour of $\boldsymbol{\psi}$ and $\boldsymbol{\psi}'$.

The crucial point in our derivation is provided by the following theorem.

Theorem 3. Let there exist an operator H such that

$$L'H - HL = \boldsymbol{V}(n, t)E^{+} + \boldsymbol{W}(n, t)I, \tag{17}$$

where $\boldsymbol{V}$, $\boldsymbol{W}$ are two $N \times N$ matrices depending of course on the pairs $(\boldsymbol{A}, \boldsymbol{B})$, $(\boldsymbol{A}', \boldsymbol{B}')$. Define the new operator

$$H' = \Omega H + \bar{H} \tag{18}$$

where

$$\bar{H}=(\mathbf{\Pi}'(n,t)\bar{\mathbf{G}}_1(t)\mathbf{\Pi}^{-1}(n+1,t))E^++\bar{\mathbf{G}}_2(t)I,$$

$\bar{\mathbf{G}}_1$, $\bar{\mathbf{G}}_2$ being two arbitrary $N\times N$ n-independent matrices and

$$\Omega H=L'H+\mathbf{G}_1E^++\mathbf{G}_2I \tag{19}$$

where

$$\mathbf{G}_1(n,t)=\mathbf{\Pi}'(n,t)\sum_{j=n+1}^{\infty}(\mathbf{\Pi}'(j,t))^{-1}\mathbf{V}(j,t)\mathbf{\Pi}(j+1,t)\mathbf{\Pi}^{-1}(n+1,t),$$

$$\mathbf{G}_2(n,t)=-\sum_{j=n}^{\infty}\mathbf{W}(j,t).$$

Then it follows that

$$L'H'-H'L=\mathbf{V}'(n,t)E^++\mathbf{W}'(n,t)I, \tag{20}$$

$\mathbf{V}'$, $\mathbf{W}'$ being again two $N\times N$ matrices related to $\mathbf{V}$, $\mathbf{W}$ through the formula

$$\begin{pmatrix}\mathbf{V}'(n,t)\\ \mathbf{W}'(n,t)\end{pmatrix}=\Lambda\begin{pmatrix}\mathbf{V}(n,t)\\ \mathbf{W}(n,t)\end{pmatrix}+\begin{pmatrix}\bar{\mathbf{V}}(n,t)\\ \bar{\mathbf{W}}(n,t)\end{pmatrix} \tag{21}$$

where

$$\Lambda\begin{pmatrix}\mathbf{V}(n,t)\\ \mathbf{W}(n,t)\end{pmatrix}=\begin{pmatrix}\mathbf{V}(n,t)\mathbf{B}(n+1,t)+\mathbf{A}'(n,t)(\mathbf{W}(n,t)+\mathbf{W}(n+1,t))+\mathbf{B}'(n,t)\mathbf{\Sigma}(n,t)\\ \mathbf{B}'(n,t)\mathbf{W}(n,t)+\mathbf{V}(n,t)\end{pmatrix}$$

$$\left.\begin{matrix}-\mathbf{\Sigma}(n,t)\mathbf{B}(n+1,t)+\sum_{j=n}^{\infty}[\mathbf{V}(j,t)\mathbf{A}(n,t)-\mathbf{A}'(n,t)\mathbf{V}(j,t)]\\ +\mathbf{\Sigma}(n-1,t)-\mathbf{\Sigma}(n,t)+\sum_{j=n}^{\infty}[\mathbf{W}(j,t)\mathbf{B}(n,t)-\mathbf{B}'(n,t)\mathbf{W}(j,t)]\end{matrix}\right), \tag{22a}$$

$$\begin{aligned}\bar{\mathbf{V}}(n,t)&=\mathbf{A}'(n,t)\bar{\mathbf{G}}_2(t)-\bar{\mathbf{G}}_2(t)\mathbf{A}(n,t)+\mathbf{B}'(n,t)\mathbf{\Pi}'(n,t)\bar{\mathbf{G}}_1(t)\mathbf{\Pi}^{-1}(n+1,t)\\ &\quad-\mathbf{\Pi}'(n,t)\bar{\mathbf{G}}_1(t)\mathbf{\Pi}^{-1}(n+1,t)\mathbf{B}(n+1,t),\\ \bar{\mathbf{W}}(n,t)&=\mathbf{B}'(n,t)\bar{\mathbf{G}}_2(t)-\bar{\mathbf{G}}_2(t)\mathbf{B}(n,t)+\mathbf{\Pi}'(n-1,t)\bar{\mathbf{G}}_1(t)\mathbf{\Pi}^{-1}(n,t)\\ &\quad+\mathbf{\Pi}'(n,t)\bar{\mathbf{G}}_1(t)\mathbf{\Pi}^{-1}(n+1,t),\end{aligned} \tag{22b}$$

$\mathbf{\Sigma}(n,t)$ being defined as

$$\mathbf{\Sigma}(n,t)=\mathbf{\Pi}'(n,t)\sum_{j=n}^{\infty}(\mathbf{\Pi}'(j,t))^{-1}\mathbf{V}(j,t)\mathbf{\Pi}(j+1,t)\mathbf{\Pi}^{-1}(n+1,t).$$

Note that the operator Λ is exactly the one introduced by Bruschi *et al* (1981).

Introducing now, as in the previous section, the basis $\{\boldsymbol{\sigma}_\nu\}_{\nu=0,\ldots,N^2-1}$ ($\boldsymbol{\sigma}_0=I$) in the space of $N\times N$ matrices, we can give, by the following theorem, the main result of this section.

Theorem 4. Let the operator H be defined through the formula

$$H=g_{1,\nu}(\Omega,t)(\mathbf{\Pi}'(n,t)\boldsymbol{\sigma}_\nu\mathbf{\Pi}^{-1}(n+1,t))E^++g_{2,\nu}(\Omega,t)\boldsymbol{\sigma}_\nu I, \tag{23}$$

where $g_{1,\nu}$, $g_{2,\nu}$ are arbitrary entire functions of their first argument, and let φ be an

$N \times N$ matrix determined by the asymptotic behaviour of ψ and ψ'. Then

$$\psi'\varphi - H\psi = 0 \tag{24}$$

if and only if

$$g_{1,\nu}(\Lambda, t)\begin{pmatrix} B'(n,t)\Pi'(n,t)\sigma_\nu\Pi^{-1}(n+1,t) - \Pi'(n,t)\sigma_\nu\Pi^{-1}(n+1,t)B(n+1,t) \\ \Pi'(n-1,t)\sigma_\nu\Pi^{-1}(n,t) - \Pi'(n,t)\sigma_\nu\Pi^{-1}(n+1,t) \end{pmatrix}$$
$$+ g_{2,\nu}(\Lambda, t)\begin{pmatrix} A'(n,t)\sigma_\nu - \sigma_\nu A(n,t) \\ B'(n,t)\sigma_\nu - \sigma_\nu B(n,t) \end{pmatrix}. \tag{25}$$

Corollary 4.1. If the 'potentials' (A, B), (A', B') satisfy equation (31) and the eigenfunctions ψ, ψ' are characterised by the asymptotic behaviour

$$\psi(n,t;z)(\psi'(n,t;z)) \underset{n\to+\infty}{\sim} z^{-n} + z^n R(z,t)(R'(z,t)) \tag{26}$$

which implies

$$\varphi(z,t) = (zg_{1,\nu}(\lambda,t) + g_{2,\nu}(\lambda,t))\sigma_\nu,$$

then the following relationship holds between the reflection coefficients R, R':

$$\varphi(z,t)R(z,t) = R'(z,t)\varphi(z^{-1},t).$$

Hence we have recovered the whole class of BTs for the NDDEs derived by Bruschi *et al* (1981).

We end this section by the following remark. Assume that t is also a discrete variable, $t = mh$ (m is a new integer variable); then any matrix function $F(n, t)$ so far introduced can be considered as a function of two integer variables n, m:

$$F(n,t) \to F(n,m).$$

If we now define the primed quantities according to the prescription

$$F'(n,t) = F(n, m+1),$$

equation (31) provides a system of nonlinear difference–difference equations for the matrices $A(n, m)$, $B(n, m)$, which can be also solved by the inverse spectral transform, using the asymptotic relationship

$$\varphi(z,m)R(z,m) = R(z,m+1)\varphi(z^{-1},m).$$

The simplest equation of this class obtains when $g_{1,\nu}$, $g_{2,\nu}$ do not depend on z, t and reads

$$A(n,m+1) - A(n,m) = B(n,m+1)\Pi(n,m+1)\Pi^{-1}(n+1,m)$$
$$- \Pi(n,m+1)\Pi^{-1}(n+1,m)B(n+1,m),$$
$$B(n,m+1) - B(n,m) = \Pi(n-1,m+1)\Pi^{-1}(n,m) - \Pi(n,m+1)\Pi^{-1}(n+1,m).$$

3. Proofs

Rather than giving detailed proofs, we will try to elucidate the points which are essential for the derivation of the previous lemmas, theorems and corollaries.

Proof of Lemma 1. The proof is closely analogous to that given by Bruschi and Ragnisco (1980a) for the continuous counterpart of this lemma. Differentiating equation (1) with respect to t, we obtain

$$(L-\lambda)\zeta(n, t; z)=\dot{\lambda}\psi(n, t; z) \tag{27}$$

where $\zeta=\dot{\psi}+M\psi$. As $L(t)$ is a one-parameter family of spectral operators, all having the same domain $\mathscr{D}$, we can express ζ, which clearly belongs to $\mathscr{D}$, in terms of the complete set of generalised eigenfunctions $\{\psi(z')\}$. Now, equation (27) implies that, in the expansion of ζ, only the term proportional to $\psi(z)$ survives (otherwise the eigenfunctions will not be linearly independent). Hence the LHS of (27) is zero, and so must be the RHS, which implies $\dot{\lambda}=0$.

Proof of Theorem 1. It follows by direct calculation of the commutator $[L, M']$ once we have defined M' according to formulae (5)–(10).

Proof of Theorem 2. It is a trivial consequence of lemma 1 and theorem 1.

Proof of Corollary 2.1. It follows from theorem 2 after explicit evaluation of the asymptotic form of the evolution equation (12), once we have defined M according to (11).

Proof of Lemma 2. To prove that (16) implies (15), one has merely to insert into (14b) the expression of ψ' in terms of ψ given by (16), using (14a) and taking advantage of the completeness of the set $\{\psi(z')\}$. Conversely, applying (15) to a generic eigenfunction ψ and using (14a), one immediately obtains (16).

Proof of Theorem 3. It follows by direct calculation of the operator $L'H'-H'L$, taking into account formulae (17)–(22).

Proof of Theorem 4. Due to lemma 2, formula (24) holds if and only if $L'H-HL=0$. On the other hand, since now H is given by (23), taking advantage of the arbitrariness of the matrices $\bar{G}_1$, $\bar{G}_2$, it easily follows that $L'H-HL$ is just the RHS of formula (25).

Proof of Corollary 4.1. It follows by direct calculation of the asymptotic form of equation (24), H being given by (23). Of course, one has to take into account (26).

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